

Asymptotic properties of infinite directed unions of local quadratic transforms

William Heinzer^a, Bruce Olberding^b, Matthew Toeniskoetter^c

^a*Department of Mathematics, Purdue University, West Lafayette, Indiana 47907*

^b*Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003-8001*

^c*Department of Mathematics, Purdue University, West Lafayette, Indiana 47907*

Abstract

Let (R, \mathfrak{m}) be a regular local ring of dimension at least 2. For each valuation domain birationally dominating R , there is an associated sequence $\{R_n\}$ of local quadratic transforms of R . We consider the case where this sequence $\{R_n\}_{n \geq 0}$ is infinite and examine properties of the integrally closed local domain $S = \bigcup_{n \geq 0} R_n$ in the case where S is not a valuation domain. For this sequence, there is an associated boundary valuation ring $V = \bigcup_{n \geq 0} \bigcap_{i \geq n} V_i$, where V_i is the order valuation ring of R_i . There exists a unique minimal proper Noetherian overring T of S . T is the regular Noetherian UFD obtained by localizing outside the maximal ideal of S and $S = V \cap T$. In the present paper, we define functions w and e , where w is the asymptotic limit of the order valuations and e is the limit of the orders of transforms of principal ideals. We describe V explicitly in terms of w and e and prove that V is either rank 1 or rank 2. We define an invariant τ associated to S that is either a positive real number or $+\infty$. If τ is finite, then S is archimedean and T is not local. In this case, the function w defines the rank 1 valuation overring W of V and W dominates S . The rational dependence of τ over $w(T^\times)$ determines whether S is completely integrally closed and whether V has rank 1. We give examples where S is completely integrally closed. If τ is infinite, then S is non-archimedean and T is local. In this case, the function e defines the rank 1 valuation overring E of V . The valuation ring E is a DVR and E dominates T , and in certain cases we prove that E is the order valuation ring of T .

Keywords: Regular local ring, local quadratic transform, valuation ring, complete integral closure, archimedean domain, generalized Krull domain

2010 MSC: 13H05, 13A15, 13A18

Email addresses: heinzer@purdue.edu (William Heinzer), olberdin@nmsu.edu (Bruce Olberding), mtoenisk@purdue.edu (Matthew Toeniskoetter)

1. Introduction and summary

Let (R, \mathfrak{m}) be a regular local ring and let $S = \bigcup_{n \geq 0} R_n$ be an infinite directed union of local quadratic transforms as in the abstract. In [11], the authors consider ideal-theoretic properties of the integral domain S . The ring S is local and integrally closed. Abhyankar proves in [1, Lemma 12, p. 337] that if $\dim R = 2$, then S is a valuation domain. However, if $\dim S \geq 3$, then S is no longer a valuation domain in general. In the case where $\dim R \geq 3$, David Shannon examines properties of S in [21]. This motivates the following definition.

Definition 1.1. Let R be a regular local ring with $\dim R \geq 2$ and let $\{R_n\}_{n \geq 0}$ be an infinite sequence of regular local rings, where $R = R_0$ and R_{n+1} is a local quadratic transform of R_n for each $n \geq 0$. Then $S = \bigcup_{n \geq 0} R_n$ is a *Shannon extension* of R .

Let S be a Shannon extension of $R = R_0$ and let F denote the field of fractions of R . Associated to each of the regular local rings (R_i, \mathfrak{m}_i) , there is a rank 1 discrete valuation ring V_i defined by the *order function* ord_{R_i} , where for $x \in R_i$, $\text{ord}_{R_i}(x) = \sup\{n \mid x \in \mathfrak{m}_i^n\}$. The family $\{V_i\}_{i=0}^\infty$ determines a unique set

$$V = \bigcup_{n \geq 0} \bigcap_{i \geq n} V_i = \{a \in F \mid \text{ord}_{R_i}(a) \geq 0 \text{ for } i \gg 0\}.$$

The set V consists of the elements in F that are in all but finitely many of the V_i . In [11], the authors prove that V is a valuation domain that birationally dominates S , and call V the *boundary valuation ring* of the Shannon extension S .

In Section 2, we review the concept of the transform of an ideal, and in Sections 3 and 4, we discuss previous results on Shannon extensions. Theorem 3.2 describes an intersection decomposition $S = V \cap T$ of a Shannon extension S , where V is the boundary valuation of S and T is the intersection of all the DVR overrings of R that properly contain S . In Setting 3.3, we fix notation to use throughout the remainder of the paper. In Discussion 4.2, we describe conditions in order that S be a valuation domain.

In Section 5 we consider asymptotic behavior of the family $\{\text{ord}_{R_n}\}_{n \geq 0}$ of order valuations of a Shannon extension $S = \bigcup_{n \geq 0} R_n$. For nonzero $a \in S$, we fix some n such that $a \in R_n$ and define $e(a) = \lim_{i \rightarrow \infty} \text{ord}_{R_{n+i}}((aR_n)^{R_{n+i}})$, where $(aR_n)^{R_{n+i}}$ denotes the transform in R_{n+i} of the ideal aR_n . For nonzero elements $a, b \in S$, we define $e(\frac{a}{b}) = e(a) - e(b)$. In Lemmas 5.2 and 5.4, we prove the function e is well defined and that e describes factorization properties of elements in S .

We fix an element $x \in S$ such that xS is primary to the maximal ideal of S . Theorem 5.6 proves that the asymptotic limit

$$\lim_{n \rightarrow \infty} \frac{\text{ord}_n(q)}{\text{ord}_n(x)}$$

exists for every nonzero element q in the quotient field F of S , but may take values $\pm\infty$. We denote this function $w : F \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, where $w(0) = +\infty$.

Let $a \in S$ be nonzero, fix m such that $a \in R_m$, and denote $a_n R_n$ to be the transform of $a R_m$ in R_n for all $n \geq m$. Let \mathfrak{m}_n denote the maximal ideal of R_n and x_n be such that $\mathfrak{m}_n R_{n+1} = x_n R_{n+1}$. In Theorem 5.10, we prove that

$$w(a) = \sum_{n=m}^{\infty} \text{ord}_n(a_n) w(x_n).$$

This allows us to define the invariant $\tau = \sum_{n=0}^{\infty} w(x_n)$ associated with the sequence $\{R_n\}_{n \in \mathbb{N}}$. In Theorem 6.1, for S with $\dim S \geq 2$, we prove that $\tau < \infty \iff S$ is archimedean $\iff w$ defines a valuation on F that dominates $S \iff S$ is dominated by a rank 1 valuation domain.

By construction, $w(x) = 1$, so the image of w contains nonzero rational values. If the image of w also has irrational values, Proposition 5.12 gives an explicit finite upper bound for τ .

Let F^\times denote the nonzero elements in the quotient field of S . If S is archimedean with $\dim S \geq 2$ we prove in Theorem 6.4 that the function

$$\begin{aligned} v : F^\times &\rightarrow \mathbb{R} \oplus \mathbb{Z} \\ q &\mapsto (w(q), -e(q)), \end{aligned}$$

defines a valuation associated to the boundary valuation ring V of S , where $\mathbb{R} \oplus \mathbb{Z}$ is ordered lexicographically. It follows that V has either rank 1 or rank 2.

In Section 7, we consider the complete integral closure S^* of an archimedean Shannon extension S with $\dim S \geq 2$. We prove in Theorem 7.1 that the almost integral elements in F over S are precisely the elements $a \in T$ such that $w(a) = 0$ and $e(a) > 0$. Together with Theorem 5.10, this allows us to characterize in Theorem 7.4 whether S is completely integrally closed in terms of the rational dependence of τ over the subgroup $w(T^\times)$ of \mathbb{R} . If S is not completely integrally closed, we prove in Theorem 7.4 that S^* is a generalized Krull domain. In Examples 7.5 and 7.6, we describe a method to construct examples of completely integrally closed archimedean Shannon extensions that are not valuation domains.

If S is non-archimedean, we prove in Theorem 8.1 for $a \in F^\times$ that $e(a) > 0$ implies $w(a) = +\infty$ and the set $P_\infty = \{a \in S \mid w(a) = +\infty\}$ is a prime ideal of both

S and T . Let xS be primary to the maximal ideal of S and denote $P = \bigcap_{n \geq 0} x^n S$. For a Shannon extension S with $\dim S \geq 2$, we prove in Theorem 8.3 that S is non-archimedean $\iff T = (P :_F P) \iff P \neq (0) \iff$ every nonmaximal prime ideal of S is contained in $P \iff T$ is a local ring $\iff T$ is the complete integral closure of S . If S is non-archimedean, we prove in Theorem 8.5 that e defines a DVR, w induces a rational rank 1 valuation on T/P_∞ , and V is the composite valuation ring of e and w .

In general, our notation is as in Matsumura [16]. Thus a local ring need not be Noetherian. An element x in the maximal ideal \mathfrak{m} of a regular local ring R is said to be a *regular parameter* if $x \notin \mathfrak{m}^2$. It then follows that the residue class ring R/xR is again a regular local ring. We refer to an extension ring B of an integral domain A as an *overring* of A if B is a subring of the quotient field of A . If, in addition, A and B are local and the inclusion map $A \hookrightarrow B$ is a local homomorphism, we say that B *birationally dominates* A . We use UFD as an abbreviation for unique factorization domain, and DVR as an abbreviation for rank 1 discrete valuation ring. For the definition of a local quadratic transform, see [1, Definition 3] or [15].

We thank Alan Loper and Hans Schoutens for correspondence about infinite directed unions of local quadratic transformations, and for their collaboration in the article [11].

2. Transform of an ideal

The concept of the transform of an ideal is used extensively in [11]. In this article we often deal with transforms of principal ideals. Properties of the transform of an ideal are given in Definition 2.1 and Remark 2.2.

Definition 2.1. Let $A \subseteq B$ be Noetherian UFDs with B an overring of A , and let I be a nonzero ideal of A . The ideal I can be written uniquely as $I = P_1^{e_1} \cdots P_n^{e_n} J$, where the P_i are principal prime ideals of A , the e_i are positive integers and J is an ideal of A not contained in a principal ideal of A [15, p. 206]. For each i , set $Q_i = P_i(A \setminus P_i)^{-1} B \cap R$. If $B \subseteq A_{P_i}$, then $A_{P_i} = B_{Q_i}$, and otherwise $Q_i = B$. The *transform*¹ of I in B is the ideal

$$I^B = Q_1^{e_1} \cdots Q_n^{e_n} (JB)(JB)^{-1},$$

where $(JB)^{-1}$ is the fractional B -ideal consisting of all elements x in the quotient field of B for which $xJB \subseteq B$. Alternatively, $I^B = Q_1^{e_1} \cdots Q_n^{e_n} K$, where K is the

¹The terminology used by Granja in [5, p. 1349] is strict transform for what Lipman calls the transform.

unique ideal of B such that $JB = xK$ for some $x \in B$ and B -ideal K not contained in a proper principal ideal of B .

Lipman [15, Lemma 1.2 and Proposition 1.5] establishes the following results about transforms:

Remark 2.2. Let $A \subseteq B \subseteq C$ be Noetherian UFDs with B and C overrings of A . Then

- (1) $(I^B)^C = I^C$ for all nonzero ideals I of A .
- (2) $(IJ)^B = I^B J^B$ for all nonzero ideals I and J of A .
- (3) Let P be a nonzero principal prime ideal of A . Then the following are equivalent.
 - (a) $P^B \neq B$.
 - (b) $B \subseteq A_P$.
 - (c) $PB \cap A = P$.
 - (d) $P^B = Q$ is a prime ideal of B such that $Q \cap A = P$.
 - (e) $P^B = Q$ is a prime ideal of B and $A_P = B_Q$.

Specializing to the case in which R is a regular local ring, we obtain an explicit formula for the transform of an ideal of R in the regular local rings of a sequence of local quadratic transforms of R . A proof for Item 3 of Remark 2.3 is given in [9, Lemma 3.6 and Remark 3.7].

Remark 2.3. Let $\{(R_i, \mathfrak{m}_i)\}_{i=0}^\infty$ be a sequence of local quadratic transforms of d -dimensional regular local rings with $d \geq 2$. For each i , let x_i be an element of \mathfrak{m}_i such that $\mathfrak{m}_i R_{i+1} = x_i R_{i+1}$. Let I be an ideal in R_0 .

- (1) $IR_1 = \mathfrak{m}_0^{\text{ord}_{R_0}(I)} IR_1 = x_0^{\text{ord}_{R_0}(I)} IR_1$ and $I^{R_1} = x_0^{-\text{ord}_{R_0}(I)} IR_1$.²
- (2) For $n \geq 0$,

$$IR_n = \left(\prod_{i=0}^{n-1} x_i^{\text{ord}_{R_i}(I^{R_i})} \right) I^{R_n} = \left(\prod_{i=0}^{n-1} \mathfrak{m}_i^{\text{ord}_{R_i}(I^{R_i})} \right) I^{R_n}.$$

- (3) The sequence of nonnegative integers $\text{ord}_{R_i}(I^{R_i})$ is nonincreasing.

²In [5, p. 1349], this last equation is used to define the (strict) transform of a height 1 prime ideal in R_1 .

3. Shannon extensions

In this section we establish the setting in which we work throughout the rest of the paper. We first recall essential results from [11]. Let S be a Shannon extension of a regular local ring R . The boundary valuation ring V of S is given by

$$V = \{a \in F \mid \text{ord}_{R_i}(a) \geq 0 \text{ for } i \gg 0\}.$$

The valuation ring V is the unique boundary point for the set of order valuation rings of the R_i with respect to the patch topology on the space of valuation overrings of R ; see [11, Section 5]. Existence and uniqueness of V is a consequence of the following lemma.

Lemma 3.1. [11, Lemma 5.2] *Let $(R, \mathfrak{m}) = (R_0, \mathfrak{m}_0)$ be a regular local ring and let $S = \bigcup_{i \geq 0} R_i$ be a Shannon extension of R . For each nonzero element a in the quotient field of R , precisely one of following hold: either $\text{ord}_{R_i}(a) > 0$ for $i \gg 0$, $\text{ord}_{R_i}(a) = 0$ for $i \gg 0$, or $\text{ord}_{R_i}(a) < 0$ for $i \gg 0$.*

In addition to the boundary valuation ring V , we work extensively with the Noetherian hull T of S . The authors establish in [11, Theorems 4.1 and 5.4] basic properties of T and demonstrate the intersection decomposition $S = V \cap T$.

Theorem 3.2. [11, Theorems 4.1 and 5.4] *Let S be a Shannon extension of a regular local ring R . Let N be the maximal ideal of S , let T be the intersection of all the DVR overrings of R that properly contain S , and let V be the boundary valuation ring of S . Then:*

- (1) $S = V \cap T$.
- (2) *There exists $x \in N$ such that xS is N -primary, and $T = S[1/x]$ for any such x . It follows that the units of T are precisely the ratios of N -primary elements of S and $\dim T = \dim S - 1$.*
- (3) *T is a localization of R_i for $i \gg 0$. In particular, T is a Noetherian regular UFD.*
- (4) *T is the unique minimal proper Noetherian overring of S .*

To simplify hypotheses, we establish a setting for the rest of the article.

Setting 3.3. We make the following assumptions throughout the paper.

- (1) R is a regular local ring with maximal ideal \mathfrak{m} and quotient field F .
- (2) $\{R_n\}_{n \geq 0}$ is an infinite sequence of regular local rings, where $R = R_0$ and R_{n+1} is a local quadratic transform of R_n for each $n \geq 0$.

- (3) $\dim R = \dim R_n = d \geq 2$ for all $n \geq 0$. Since Krull dimension does not increase upon taking local quadratic transform, we achieve this condition by replacing R with R_n for some large n .
- (4) $S = \bigcup_{n=0}^{\infty} R_n$ denotes the Shannon extension of R along $\{R_n\}$ and $N = \bigcup_{n=0}^{\infty} \mathfrak{m}_n$ denotes the maximal ideal of S .
- (5) For each $n \geq 0$, denote by $\text{ord}_n : F \rightarrow \mathbb{Z} \cup \{\infty\}$ the order valuation of R_n and by $V_n = \{q \in F : \text{ord}_n(q) \geq 0\}$ the corresponding DVR.
- (6) Fix $x \in S$ such that xS is N -primary and denote by $T = S[1/x]$ the Noetherian hull of S ; see Theorem 3.2(2).
- (7) T is a localization of R . By Theorem 3.2(2), we achieve this by replacing R with R_n for some large n .
- (8) V denotes the boundary valuation ring for S and S^* denotes the complete integral closure of S .

Setting 3.3(7) is equivalent to for all $n \geq 0$, $\mathfrak{m}_n T = T$. This, together with Remark 2.3, implies the following fact:

Remark 3.4. Assume Setting 3.3 and let $I \subset R_n$ be an ideal. Then for $m \geq n$, Setting 3.3(7) implies that $(I^{R_m})T = IT = I^T$.

We separate Shannon extensions into those that are archimedean and those that are non-archimedean, where we use the following definition.

Definition 3.5. An integral domain A is *archimedean* if $\bigcap_{n>0} a^n A = 0$ for each nonunit $a \in A$.

Remark 3.6. If A is a non-archimedean integral domain, then $\dim A \geq 2$. Indeed, if there exists a nonzero nonunit $a \in A$ such that $0 \neq b \in \bigcap_{i>0} a^i A$ for some $b \in A$, then a maximal ideal containing a cannot be a minimal prime of bA . A Shannon extension S of R as in Setting 3.3 is archimedean if and only if $\bigcap_{n>0} x^n S = 0$, where x is as in Setting 3.3(5). A Shannon extension S with $\dim S = 1$ is a rank 1 valuation ring, cf. [11, Theorem 8.1].

A Shannon extension S is a directed union of integrally closed domains, and is therefore integrally closed. However, there often exist elements in the field F that are almost integral over S and not in S . If this happens, then S is not completely integrally closed;³ see for example Theorem 7.2. The complete integral closure S^* of an archimedean Shannon extension S is described by the following theorem.

³An element θ in the field of fractions of an integral domain A is *almost integral* over A if the ring $A[\theta]$ is a fractional ideal of A . The integral domain A is *completely integrally closed* if each element in the field of fractions of A that is almost integral over A is already in A . The *complete integral closure* of a domain is the ring of almost integral elements in its field of fractions. In general, the complete integral closure of a domain may fail to be completely integrally closed.

Theorem 3.7. [11, Theorem 6.2] *Assume notation as in Setting 3.3 and assume S is archimedean. Denote by W the rank one valuation overring of the boundary valuation ring V . Then:*

- (1) $S^* = N :_F N = W \cap T$.
- (2) $S = S^* \iff V = W$.
- (3) *If $S \neq S^*$, then S^* is a generalized Krull domain, and W is the unique rank 1 valuation overring \mathcal{V} of S such that $S^* = T \cap \mathcal{V}$.*

4. Essential prime divisors of a Shannon extension

Definition 4.1. For an integral domain A , let

$$\text{epd}(A) = \{A_P \mid P \text{ is a height 1 prime ideal of } A\}.$$

The notation is motivated by the fact that if A is a Noetherian integrally closed domain, then $\text{epd}(A)$ is the set of essential prime divisors of A . With notation as in Setting 3.3, define

$$\text{epd}(S/R) = \{\mathcal{V} \in \bigcup_{i \geq 0} \text{epd}(R_i) \mid S \subseteq \mathcal{V}\}.$$

Discussion 4.2. The authors show in [11, Remark 2.4 and Lemma 3.2] that the set $\text{epd}(S/R)$ consists of the essential prime divisors of R that contain S along with the order valuation rings of any of the R_i that contain S . Moreover, S is a rank 1 valuation domain if and only if $\text{epd}(S/R) = \emptyset$ [21, Proposition 4.18],⁴ and S is a rank 2 valuation domain if and only if $\text{epd}(S/R)$ consists of a single element [4, Theorem 13].⁵

A Shannon extension S is a rank 1 valuation domain if and only if $\dim S = 1$, cf. [11, Theorem 8.1]. If S is not a rank 1 valuation domain, then $\dim S \geq 2$ and $\text{epd}(S/R) = \text{epd}(S)$. In this case, Granja and Sanchez-Giralda [6, Definition 3] define $\{R_i\}$ to be a *quadratic sequence along a prime ideal \mathfrak{p} of R* if the transform of \mathfrak{p} in R_i is a proper ideal of R_i for all i , or equivalently, $S \subseteq R_{\mathfrak{p}}$ [6, Remark 4].

Let T be the Noetherian hull of S . Theorem 3.2(2) implies that $\text{epd}(S/R) = \text{epd}(T)$. In addition, we have

- (1) $\{R_i\}$ is a quadratic sequence along \mathfrak{p} if and only if $T \subseteq R_{\mathfrak{p}}$, and
- (2) \mathfrak{p} is maximal for the sequence $\{R_i\}$ as in [6, Definition 6] if and only if $\mathfrak{p}R_{\mathfrak{p}} \cap T$ is a maximal ideal of T .

⁴In this case, the sequence $\{R_i\}$ is said to switch strongly infinitely often.

⁵In this case, the sequence $\{R_i\}$ is said to be height 1 directed.

Assume $\{R_i\}$ is a quadratic sequence along \mathfrak{p} and R/\mathfrak{p} is regular. Let $\mathfrak{p}_i = \mathfrak{p}R_{\mathfrak{p}} \cap R_i$ denote the transform of \mathfrak{p} in R_i . Granja and Sanchez-Giralda [6, Theorem 8] prove that \mathfrak{p} is maximal for the sequence $\{R_i\}$ if and only if $S/(\mathfrak{p}R_{\mathfrak{p}} \cap S) = \bigcup_{i \geq 0} R_i/\mathfrak{p}_i$ is a rank 1 valuation domain.

Definition 4.3. Assume Setting 3.3 and let $p \in R_i$ be a nonzero prime element. We call p an *essential prime element* of S/R_i if $(R_i)_{pR_i} \in \text{epd}(S/R)$.

If p is an essential prime element of S/R_i , then it follows from results described in Theorem 3.2(2) and Discussion 4.2 that pT is a height 1 prime ideal of T . Notice, however, that $p \notin R_j$ for $j < i$, and p is not a prime element in R_j for $j > i$. The ideal pS is a proper ideal of S , but is not a prime ideal of S .

Proposition 4.4. Assume Setting 3.3.

- (1) Let p be a prime element of R_n . Then p is an essential prime element of S/R_n $\iff (pR_n)^{R_m} \neq R_m$ for all $m > n \iff p \notin T^\times$.
- (2) Let $a \in R_n$. Then $a \in T^\times$ if and only if $(aR_n)^{R_m} = R_m$ for $m \gg n$.
- (3) Let $a \in R_n$ be a nonzero nonunit. Then $a = u\tilde{a}$ in R_n , where $u \in R_n \cap T^\times$ and \tilde{a} is a possibly empty product of essential prime elements of S/R_n . By convention, an empty product is 1.
- (4) Let a be a nonzero nonunit in R_n and as in (3) write $a = up_1 \cdots p_n$, where $u \in T^\times$ and p_1, \dots, p_n are essential prime elements of S/R_n . For each i , let $P_i = p_i R_n$. Then for each $m \geq n$, we have $(aR_m)^{R_i} = P_1^{R_i} \cdots P_r^{R_i} R_i$ for $i \gg m$.

Proof. The first equivalence of item (1) follows from Remark 2.2(3). To see equivalence with the third statement, use Remark 3.4 for the “ \Leftarrow ” implication. The “ \Rightarrow ” implication follows from the fact that if p is an essential prime element of S/R_n , then pT is a height 1 prime ideal of T .

To see Item (2), let $a \in R_n$. Since the cases where $a = 0$ or a is a unit are trivial, we may assume a is a nonzero nonunit in R_n . Since R_n is a UFD, we may write $a = p_1 \cdots p_n$, where the p_i are prime elements of R_n . Then from Item (1) and Remark 2.2(2), it follows that a is a unit in $T \iff$ each p_i is a unit in $T \iff (p_i R_n)^{R_m} = R_m$ for $m \gg n$ for all $i \iff (aR_n)^{R_m} = R_m$ for $m \gg n$.

Item (3) follows from (1) and the fact that R is a UFD.

For item (4), from Remark 2.3 we have that $aR_m = uu'P_1^{R_m} \cdots P_r^{R_m}$, where $u'R_m$ is a product of powers of $\mathfrak{m}_k R_m, \dots, \mathfrak{m}_{m-1} R_m$. The assumption in Setting 3.3(7) implies that $\mathfrak{m}_k S$ is N -primary for each k , so $u' \in T^\times$. Moreover, for $i \geq m$, Remark 2.2(2) implies that $(aR_m)^{R_i} = (uu'R_m)^{R_i} P_1^{R_i} \cdots P_r^{R_i}$. Since $u, u' \in T^\times$, it follows from (2) that $(uu'R_m)^{R_i} = R_i$ for $i \gg m$. Hence $(aR_m)^{R_i} = P_1^{R_i} \cdots P_r^{R_i} R_i$ for $i \gg m$. \square

5. Asymptotic behavior of the order valuations

Assume notation as in Setting 3.3, so in particular, fix $x \in S$ such that xS is N -primary. In this section we analyze the limit

$$\lim_{n \rightarrow \infty} \frac{\text{ord}_n(a)}{\text{ord}_n(x)} \quad (1)$$

for nonzero elements $a \in F$. This limit plays a key role in our description of Shannon extensions. If S is an archimedean Shannon extension, we show in Section 6 that the limit given in Equation 1 defines the rank 1 valuation overring of the boundary valuation ring of S . If S is a non-archimedean Shannon extension, we show in Section 8 that the limit given in Equation 1 induces a rational rank 1 valuation on a certain homomorphic image S/P of S .

For an ideal I of R , Remark 2.3(3) implies the sequence of nonnegative integers $\text{ord}_{R_i}(I^{R_i})$ is nonincreasing and thus must converge. We use the following definition.

Definition 5.1. Assume Setting 3.3 and let $a \in S$ be nonzero. Then $a \in R_n$ for some $n \geq 0$. Define $e(a) = \lim_{i \rightarrow \infty} \text{ord}_{n+i}((aR_n)^{R_{n+i}})$.

For $\frac{a}{b} \in F$, where a, b are nonzero elements in S , let $n \in \mathbb{N}$ be such that $a, b \in R_n$ and define $e(\frac{a}{b}) = e(a) - e(b)$.

For nonzero $a \in S$, $e(a)$ is a finite non-negative integer. A priori, $e(a)$ depends on the starting point R_n and $e(\frac{a}{b})$ depends on the representation of $\frac{a}{b}$ as an element in F . In Lemma 5.2, we prove $e(-)$ is independent of starting point and representation.

Lemma 5.2. Assume Setting 3.3. Let e be as in Definition 5.1. Then:

- (1) For nonzero $a \in F$, $e(a)$ is well defined.
- (2) For each nonzero $a \in R_n \setminus T^\times$, there exists a factorization $aR_n = u p_1 \cdots p_r$ in R_n , where $u \in R_n \cap T^\times$ and p_1, \dots, p_r are not necessarily distinct essential prime elements of S/R_n . Then $e(a) = \sum_{i=1}^r e(p_i)$.
- (3) For nonzero $a, b \in F$, $e(ab) = e(a) + e(b)$.
- (4) For nonzero $a \in S$, $e(a) = 0$ if and only if $a \in T^\times$.

Proof. By Proposition 4.4(2) and (4), for each $a \in S$, $e(a)$ is independent of the starting point R_n . Let $a \in S \setminus T^\times$ with a nonzero. Proposition 4.4(3) implies the factorization $a = u\tilde{a} = u p_1 \cdots p_r$ in R_n as in the statement of (2), so we have $aR_n = u p_1 \cdots p_r R_n$. By Proposition 4.4(4), $e(a) = e(p_1 \cdots p_r) = e(p_1) + \cdots + e(p_r)$, where the latter assertion follows from the fact that $\text{ord}_i(a) = \text{ord}_i(p_1) + \cdots + \text{ord}_i(p_r)$ for all i . This verifies (2). Item (3), as well as the fact that $e(a)$ is well defined for all $a \in F$, now follow from (2) and the fact that R_n is a UFD. Item (4) is a consequence of Proposition 4.4(2). \square

Remark 5.3. Assume Setting 3.3. From Lemma 5.2(2) it follows that $\text{epd}(S/R) = \emptyset$ if and only if $e(a) = 0$ for all nonzero $a \in S$. Thus as in Discussion 4.2, $e(a) = 0$ for all nonzero $a \in S$ if and only if S is a rank 1 valuation domain.

Lemma 5.4. Assume Setting 3.3 and let $a, b \in S$ be nonzero. For $n \gg 0$, in R_n , there exist factorizations $a = u\tilde{a}$ and $b = v\tilde{b}$, where $u, v \in R_n \cap T^\times$, \tilde{a}, \tilde{b} are products of essential prime elements of S/R_n , $\text{ord}_n(\tilde{a}) = e(a)$ and $\text{ord}_n(\tilde{b}) = e(b)$. Furthermore, for any such factorization,

- (1) If $\text{ord}_n(a) \geq \text{ord}_n(b)$ for $n \gg 0$, then v divides u in R_n for $n \gg 0$.
- (2) If $\text{ord}_n(a) = \text{ord}_n(b)$ for $n \gg 0$, then $vR_n = uR_n$ for $n \gg 0$, and $e(a) = e(b)$.

Proof. By replacing R with R_n for some sufficiently large n , we may assume that $a, b \in R$. As in Proposition 4.4(3), we may write $a = u\tilde{a}$ and $b = v\tilde{b}$, where \tilde{a}, \tilde{b} are the products of essential primes of S/R and $u, v \in R \cap T^\times$. By again replacing R with R_n for sufficiently large n , we may assume that $\text{ord}_0(\tilde{a}) = e(a)$ and $\text{ord}_0(\tilde{b}) = e(b)$ as in Lemma 5.2(2).

(1) Assume that $\text{ord}_n(a) \geq \text{ord}_n(b)$ for $n \gg 0$. By factoring out their greatest common divisor in R , we may assume a, b are relatively prime in R . It suffices to show v is a unit in R . We proceed as in the proof of [11, Lemma 5.2]. Denote $a_0 = a$, $b_0 = b$, $Q_0 = (a_0, b_0)R_0$, and let $Q_i = (a_i, b_i)$ be the transform of Q_0 in R_i , so $Q_i = \mathfrak{m}_i^{\text{ord}_i Q_i} Q_{i+1}$. Let $e = \lim_{n \rightarrow \infty} \text{ord}_n(Q_n)$. Since $\text{ord}_n(a) \geq \text{ord}_n(b)$ for $n \gg 0$, we have $e = \lim_{n \rightarrow \infty} \text{ord}_n(b_n)$. By replacing R with R_n for some $n \gg 0$, we may assume that for all $i \geq 0$, $\text{ord}_i(b_i) = e$. Thus $b_i R_i$ is the transform in R_i of the principal ideal $b_0 R_0$.

Consider the factorization $b = v\tilde{b}$ as above. Let $v_i R_i, \tilde{b}_i R_i$ be the transforms in R_i of $vR_0, \tilde{b}R_0$, respectively, so that $b_i R_i = v_i \tilde{b}_i R_i$. By Remark 2.3(3), $\text{ord}_i(\tilde{b}_i)$ and $\text{ord}_i(v_i)$ are both nonincreasing. Since $v \in T^\times$, we have $v_i R_i = R_i$ for $i \gg 0$, and $\text{ord}_i(v_i) = 0$ for $i \gg 0$. Thus $\text{ord}_i(\tilde{b}_i) = e$ for $i \gg 0$. Since $e \geq \text{ord}_0(\tilde{b}) \geq \text{ord}_i(\tilde{b}_i) = e$, we conclude that $\text{ord}_0(v) = 0$; that is, v is a unit in R , and thus divides u in R .

(2) This follows by applying (1) twice. \square

Corollary 5.5. Assume Setting 3.3. If $a \in F^\times$ is such that $e(a) = 0$, then there exists $u \in T^\times$ such that $\text{ord}_n(a) = \text{ord}_n(u)$ for all $n \gg 0$.

Proof. Write $a = \frac{b}{c}$ with $b, c \in S$ and apply Lemma 5.4 to obtain $b = u\tilde{b}$ and $c = v\tilde{c}$. Then $\frac{u}{v} \in T^\times$ and $\text{ord}_n(\frac{u}{v}) = \text{ord}_n(a)$ for all $n \gg 0$. \square

We prove in Theorem 5.6 that the limit described in Equation 1 exists.

Theorem 5.6. Assume notation as in Setting 3.3. For $a \in F^\times$ the (possibly infinite) limit

$$\lim_{n \rightarrow \infty} \frac{\text{ord}_n(a)}{\text{ord}_n(x)}$$

exists.

Proof. We construct a Dedekind cut as follows:

$$A = \left\{ \frac{p}{q} \in \mathbb{Q} \mid \frac{\text{ord}_n(a)}{\text{ord}_n(x)} \geq \frac{p}{q} \text{ for } n \gg 0 \right\}.$$

$$B = \left\{ \frac{p}{q} \in \mathbb{Q} \mid \frac{\text{ord}_n(a)}{\text{ord}_n(x)} < \frac{p}{q} \text{ for } n \gg 0 \right\}.$$

For $p/q \in \mathbb{Q}$ (assuming $q > 0$), $p/q \in A$ is equivalent to $\text{ord}_n(a^q) \geq \text{ord}_n(x^p)$ for $n \gg 0$, and $p/q \in B$ is equivalent to $\text{ord}_n(a^q) < \text{ord}_n(x^p)$ for $n \gg 0$. By the construction of A and B , it follows that for $r \in A$, $r < s$ for all $s \in B$. By Lemma 3.1, we conclude that $A \cup B = \mathbb{Q}$. Thus the limit in the statement of the theorem is equal to $\sup A = \inf B$. \square

In view of Theorem 5.6, we define a function w as follows:

Notation 5.7. Assume Setting 3.3 and define $w : F \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ by defining $w(0) = +\infty$, and for each $q \in F^\times$,

$$w(q) = \lim_{n \rightarrow \infty} \frac{\text{ord}_n(q)}{\text{ord}_n(x)}.$$

Remark 5.8. Assume Setting 3.3. Since w is the limit of valuations, w behaves like a valuation. In particular, for elements $a, b \in F$, we have:

- (1) $w(a+b) \geq \min\{w(a), w(b)\}$, and $w(a+b) = \min\{w(a), w(b)\}$ if $w(a) \neq w(b)$.
- (2) $w(ab) = w(a) + w(b)$, except in the case where one value is $+\infty$ and the other is $-\infty$.
- (3) If $a \neq 0$, then $w(a) = -w(\frac{1}{a})$.
- (4) If A is a subring of F such that $w(a) \neq -\infty$ for each $a \in A$, then

$$P = \{a \in A \mid w(a) = +\infty\}$$

is a prime ideal of A . If in addition there exists a nonzero $a \in A$ with $w(a) \neq 0$, then w induces a rank 1 valuation w' on the quotient field of A/P such that $w(a) = w'(a')$, where a' is the image of a in A/P .

Proof. Items 1, 2, and 3 follow from the fact that w is a limit of order valuations. For item 4, see [2, Remark 2, p. 387 and Prop. 4, p. 388]. Since a' is the coset $a + P$ and the elements in P have w value $+\infty$, we have $w(a) = w'(a')$. \square

We establish the basic properties of w with respect to the Shannon extension S .

Theorem 5.9. *Assume notation as in Setting 3.3.*

- (1) *If $a \in V$, then $w(a) \geq 0$.*
- (2) *If $a \in F$, then $w(a) > 0$ implies that $a \in \mathfrak{m}_V$.*
- (3) *If $a \in S$, then $w(a) > 0$ if and only if $a \in N (= \mathfrak{m}_V \cap S)$.*
- (4) *Let $a \in F^\times$ be such that $e(a) = 0$. Then $w(a)$ is finite, and $a \in V$ if and only if $w(a) \geq 0$.*
- (5) *For each $n \in \mathbb{N}$ and element $z \in \mathfrak{m}_n$, we have*

$$zR_{n+1} = \mathfrak{m}_n R_{n+1} \iff w(z) = \min\{w(y) \mid y \in \mathfrak{m}_n\}.$$

To summarize items 1, 2, and 3,

$$\{a \in F \mid w(a) > 0\} \subseteq \mathfrak{m}_V \subseteq V \subseteq \{a \in F \mid w(a) \geq 0\}$$

$$N = \{a \in S \mid w(a) > 0\}.$$

Proof. For item 1, if $a \in V$, then $\text{ord}_n(a) \geq 0$ for $n \gg 0$, so $w(a) \geq 0$. For item 2, if $a \in F$ is such that $w(a) > 0$, then $\text{ord}_n(a) > 0$ for $n \gg 0$, so $a \in \mathfrak{m}_V$.

To see item 3, let $a \in S$. The “only if” implication follows item 2. To see the “if” implication, assume that $a \in N$. Since the ideal xS is N -primary, there is a positive integer r such that $a^r \in xS$, so $a^r/x \in R_n$ for $n \gg 0$. Then $w(a^r/x) \geq 0$, so $rw(a) > w(x) = 1$, so $w(a) > \frac{1}{r} > 0$. This proves item 3.

To see item 4, let $a \in F^\times$ be such that $e(a) = 0$. By Corollary 5.5, there exists $y \in T^\times$ such that $\text{ord}_n(a) = \text{ord}_n(y)$ for $n \gg 0$, so $w(a) = w(y)$ and $a \in V$ if and only if $y \in V$. Thus to show item 4, we may assume $a = y \in T^\times$. Theorem 3.2(2) implies that $a = u/v$, where $u, v \in N$ are N -primary elements of S .

By item 3, $w(u) > 0$. Since uS is N -primary, there exists a positive integer s such that $x^s \in uS$, so by the same argument as in item 3, $w(u) \leq s$. Since $w(u)$ is positive and bounded, it is finite. Similarly $w(v)$ is finite, so we conclude that $w(a) = w(u) - w(v)$ is finite.

The principal N -primary ideals are linearly ordered as a set under inclusion [11, Corollary 5.5], so the ideals uS and vS are comparable by inclusion. From the multiplicativity of w and finiteness of $w(u)$ and $w(v)$, we conclude that $a \in S$ if and only if $w(a) \geq 0$. Since $a \in T^\times$ and $S = T \cap V$, it follows that $a \in V$ if and only if $w(a) = 0$. This completes the proof of item 4.

For item 5, let $z \in \mathfrak{m}_n$. Then $z \in x_n R_{n+1} = \mathfrak{m}_n R_{n+1}$, so we may write $z = x_n a$ for some $a \in R_{n+1}$, where $w(z) = w(x_n) + w(a)$. We have $w(a) \geq 0$ by item 1, so $w(z) \geq w(x_n)$, where equality holds if and only if $w(a) = 0$. Thus $w(x_n) = \min\{w(y) \mid y \in \mathfrak{m}_n\}$. Furthermore, item 3 implies that a is a unit in R_{n+1} if and only if $w(a) = 0$. We conclude that $zR_{n+1} = \mathfrak{m}_n R_{n+1}$ if and only if $w(z) = \min\{w(y) \mid y \in \mathfrak{m}_n\}$ \square

The function w can yield infinite values even restricted to nonzero elements of S . We prove in Theorem 6.1 that if $\dim S \geq 2$, then S is archimedean if and only if w takes only finite values on nonzero elements of F , in which case w is a valuation.

In Theorem 5.10, we give an alternate interpretation of the restriction of the function w to S .

Theorem 5.10. *Assume Setting 3.3. Let $a \in S$ be nonzero. Then $a \in R_m$ for some $m \geq 0$. Let $a_i R_i$ be the transform of $a R_m$ in R_i for all $i \geq m$. We have*

$$w(a) = \sum_{n=m}^{\infty} \text{ord}_n(a_n) w(x_n)$$

where $\mathfrak{m}_n R_{n+1} = x_n R_{n+1}$.

Proof. By Theorem 5.6, the possibly infinite limit exists. From Setting 3.3(7), we have $x_n \in T^\times$ for all $n \geq 0$.

For $n \geq m$, using Remark 2.3, we have $a R_n = \left(\prod_{i=m}^{n-1} x_i^{\text{ord}_i(a_i)} \right) a_n R_n$. Thus for all $n \geq m$ and for all $j \geq 0$,

$$\text{ord}_j(a) = \left(\sum_{i=m}^{n-1} \text{ord}_i(a_i) \text{ord}_j(x_i) \right) + \text{ord}_j(a_n).$$

Dividing both sides by $\text{ord}_j(x)$,

$$\frac{\text{ord}_j(a)}{\text{ord}_j(x)} = \left(\sum_{i=m}^{n-1} \frac{\text{ord}_i(a_i) \text{ord}_j(x_i)}{\text{ord}_j(x)} \right) + \frac{\text{ord}_j(a_n)}{\text{ord}_j(x)}.$$

Taking the limit as $j \rightarrow \infty$ on both sides and applying Theorem 5.6 on the middle terms,

$$\lim_{j \rightarrow \infty} \frac{\text{ord}_j(a)}{\text{ord}_j(x)} = \left(\sum_{i=m}^{n-1} \text{ord}_i(a_i) w(x_i) \right) + \lim_{j \rightarrow \infty} \frac{\text{ord}_j(a_n)}{\text{ord}_j(x)}.$$

So it follows that,

$$\lim_{j \rightarrow \infty} \frac{\text{ord}_j(a)}{\text{ord}_j(x)} \geq \sum_{i=m}^{\infty} \text{ord}_i(a_i) w(x_i).$$

Let $\sigma := \sum_{i=m}^{\infty} \text{ord}_i(a_i) w(x_i)$. If $e(a) = 0$, then $\text{ord}_i(a_i) = 0$ for $i \gg 0$, so that the sum is finite and the proof is complete by additivity of w as in Remark 5.8. If $\sigma = \infty$, the limit is $+\infty$ and there is nothing to show. Assume that $\sigma < \infty$ and $e(a) > 0$. Let p/q be any rational number such that the limit in the left hand side of the above inequality is greater than p/q . Then for $n \gg 0$, $\text{ord}_n(a^q) > \text{ord}_n(x^p)$. By Lemma 5.4,

since $e(x^p) = 0$, it follows that x^p divides a^q in R_n for $n \gg 0$. But for $n \gg 0$, a_n is a product of essential prime elements by Proposition 4.4(4), so that a_n, x have no common factors in R_n . Thus since by Remark 2.3(2), $aR_n = \left(\prod_{i=m}^{n-1} x_i^{\text{ord}_i(a_i)}\right) a_n R_n$, x^p divides $(\prod_{i=m}^{n-1} x_i^{\text{ord}_i(a_i)})^q$, so $p = w(x^p) < q\sigma$. Hence $\frac{p}{q} < \sigma$, and this completes the proof of the theorem. \square

In view of Theorem 5.10, we single out an invariant τ naturally associated with the sequence $\{R_n\}_{n=0}^\infty$.

Notation 5.11. With w as in Notation 5.7, we define $\tau = \sum_{n=0}^\infty w(x_n)$ where $\mathfrak{m}_n R_{n+1} = x_n R_{n+1}$ for $n \geq 0$.

The invariant τ relates the function w to the function e , c.f. Remark 6.3. In Section 6, we prove for a Shannon extension S of dimension at least 2 that τ is finite if and only if S is archimedean. In Section 7, we use τ to determine whether S is completely integrally closed in the archimedean case.

By construction, $w(x) = 1$, so the image of w contains rational values. In the case where w also takes finite irrational values, the following proposition exhibits an explicit upper bound for τ .

Proposition 5.12. *Assume Setting 3.3. Let $y_1, \dots, y_r \in \mathfrak{m}$, where $r \geq 2$. If $w(y_1), \dots, w(y_r)$ are finite and rationally independent, then*

$$\tau \leq \frac{w(y_1) + \dots + w(y_r)}{r - 1}.$$

Proof. We argue as in the proof of [10, Prop. 7.3]. We inductively prove that for all $n \geq 0$, there are elements $y_1^{(n)}, \dots, y_r^{(n)} \in \mathfrak{m}_n$ such that $w(y_1^{(n)}), \dots, w(y_r^{(n)})$ are rationally independent and

$$(r - 1) \left(\sum_{i=0}^{n-1} w(x_i) \right) + \sum_{j=1}^r w(y_j^{(n)}) \leq \sum_{j=1}^r w(y_j). \quad (2)$$

Taking $y_j^{(0)} = y_j$, the base case $n = 0$ is clear. Assume the claim is true for n . Thus we have elements $y_j^{(n)} \in \mathfrak{m}_n$ such that Equation 2 holds.

By Remark 5.8, $z \in \mathfrak{m}_n$ has minimal w -value if and only if $zR_{n+1} = \mathfrak{m}_n R_{n+1}$. Thus the set $w(x_n), w(y_1^{(n)}), \dots, w(y_r^{(n)})$ has rational rank at least r . By re-ordering, without loss of generality $w(x_n), w(y_2^{(n)}), \dots, w(y_r^{(n)})$ are rationally independent. Set $y_1^{(n+1)} = x_n$ and set $y_j^{(n+1)} = \frac{y_j^{(n)}}{x_n}$ for $2 \leq j \leq n$. It follows that $w(y_1^{(n+1)}), \dots, w(y_r^{(n+1)})$

are rationally independent. Since $w(x_n) < w(y_j^{(n)})$ for $2 \leq j \leq n$, we have by Theorem 5.9(3) that $y_j^{(n+1)} \in \mathfrak{m}_{n+1}$ for $2 \leq j \leq n$. We also have $w(x_n) \leq w(y_1^{(n)})$. Thus,

$$\begin{aligned}
(r-1) \left(\sum_{i=0}^n w(x_i) \right) + \sum_{j=1}^r w(y_j^{(n+1)}) &= (r-1) \left(\sum_{i=0}^{n-1} w(x_i) \right) + (r-1)w(x_n) + \\
&\quad w(x_n) + \sum_{j=2}^r (w(y_j^{(n)}) - w(x_n)) \\
&\leq (r-1) \left(\sum_{i=0}^{n-1} w(x_i) \right) + \sum_{j=1}^r w(y_j^{(n)}) \\
&\leq \sum_{j=1}^r w(y_j),
\end{aligned}$$

where the last inequality is a consequence of Equation 2. The conclusion follows. \square

6. Archimedean Shannon extensions

Theorem 6.1 shows in the archimedean case that the mapping w defined in Notation 5.7 is a rank 1 valuation whose valuation ring dominates S .

Theorem 6.1. *Assume Setting 3.3 and let w and τ be as in Notation 5.7 and Notation 5.11, respectively. If $\dim S \geq 2$, then the following are equivalent.*

- (1) S is archimedean.
- (2) $w(q)$ is finite for all nonzero $q \in F$.
- (3) $w(q)$ is finite for some nonzero $q \in S \setminus T^\times$.
- (4) $\tau < \infty$.
- (5) w defines a valuation on F that dominates S .
- (6) S is dominated by a rank 1 valuation domain.⁶

Proof. (1) \Rightarrow (2) Since w is multiplicative, we may assume $q \in N$. By Theorem 5.6, the limit that defines $w(q)$ exists, so $w(q)$ is finite if the sequence $\{\text{ord}_n(q)/\text{ord}_n(x)\}_{n=1}^\infty$ is bounded. Since S is archimedean, there is some integer $m \geq 0$ such that $q \notin x^m S$.

⁶An archimedean Shannon extension S of R is often birationally dominated by infinitely many rank 1 valuation domains. This is the case in [21, Example 4.17].

Thus $q/x^m \notin S$. Since $q/x^m \in T$, and $S = V \cap T$ by Theorem 3.2(1), we conclude that $q/x^m \notin V$. Thus $\text{ord}_n(q) < \text{ord}_n(x^m)$ for $n \gg 0$, which implies that $\text{ord}_n(q)/\text{ord}_n(x) < m$ for $n \gg 0$.

(2) \Rightarrow (3) This is clear.

(3) \Rightarrow (4) Since $q \in S \setminus T^\times$, it follows from Lemma 5.2 that $e(q) > 0$, where e is as in Definition 5.1. Theorem 5.10 implies that $e(q) \sum_{n \geq m} w(x_n) \leq w(q) < \infty$ for some integer $m > 0$, so it follows that $\tau < \infty$.

(4) \Rightarrow (5) Theorem 5.10 implies that $w(q)$ is finite for all $q \in F^\times$. Thus by Remark 5.8, w defines a valuation ring that dominates S .

(5) \Rightarrow (6) Since the valuation in (5) has values in \mathbb{R} , it has rank 1, so that (6) is clear.

(6) \Rightarrow (1) Let U be a rank 1 valuation domain that dominates S , and let $x \in N$. Since $\dim U = 1$, we have $\bigcap_{n > 0} x^n S \subseteq \bigcap_{n > 0} x^n U = 0$, so (1) follows. \square

Definition 6.2. In the case where S is archimedean, we denote by W the rank 1 valuation domain defined by w . Notice that W dominates S .

Remark 6.3. Assume that S is archimedean and let $a \in F^\times$. By Theorem 5.10, there exists $y \in T^\times$ such that $w(a) = w(y) + e(a)\tau$.

Proof. By Theorem 6.1, we have $\tau < \infty$. We may assume $a \in S$. It follows that $a \in R_m$ for some $m \geq 0$. As in the proof of Theorem 5.10, let $a_i R_i$ be the transform of $a R_m$ in R_i for all $i \geq m$, and let $\mathfrak{m}_i R_{i+1} = x_i R_{i+1}$ for all $i \geq 0$. Let $k \geq m$ be such that $\text{ord}_i(a_i) = e(a)$ for all $i \geq k$. By Theorem 5.10,

$$w(a) = \sum_{n \geq m} \text{ord}_n(a_n) w(x_n).$$

Then

$$\begin{aligned} w(a) &= \sum_{n \geq 0} e(a) w(x_n) - \sum_{0 \leq n < m} e(a) w(x_n) + \sum_{m \leq n < k} (\text{ord}_n(a_n) - e(a)) w(x_n) \\ &= e(a)\tau - \sum_{0 \leq n < m} e(a) w(x_n) + \sum_{m \leq n < k} (\text{ord}_n(a_n) - e(a)) w(x_n) \\ &= e(a)\tau + w \left(\prod_{0 \leq n < m} x_n^{-e(a)} \prod_{m \leq n < k} x_n^{\text{ord}_n(a_n) - e(a)} \right). \end{aligned}$$

Then y is a finite product of integer powers of the elements $x_0, \dots, x_{k-1} \in T^\times$, and we have $w(a) = w(y) + e(a)\tau$. \square

Using the mappings w and e , Theorem 6.4 gives in the archimedean case an explicit description of the valuation associated to the boundary valuation ring.

Theorem 6.4. *Assume Setting 3.3, let V be the boundary valuation ring of S , and assume that S is archimedean and $\dim S \geq 2$. Consider the function*

$$\begin{aligned} v : F^\times &\rightarrow \mathbb{R} \oplus \mathbb{Z} \\ q &\mapsto (w(q), -e(q)), \end{aligned}$$

where $\mathbb{R} \oplus \mathbb{Z}$ is ordered lexicographically. Then v is a valuation of F that defines V . It follows that V has either rank 1 or rank 2.

Proof. From Lemma 5.2(4) and Theorem 6.1 it follows that $v(q_1 q_2) = v(q_1) + v(q_2)$ for all $q_1, q_2 \in F^\times$. To prove that v is a valuation, and that v is the valuation associated to V , it suffices to show that if $a \in V$, then $v(a) \geq 0$, and if $a \in \mathfrak{M}_V$, then $v(a) > 0$.

Let a and b be nonzero elements of S . Suppose $a/b \in V$. We prove that $v(a) \geq v(b)$. By definition of V , $\text{ord}_n(a) \geq \text{ord}_n(b)$ for $n \gg 0$. It follows that $w(a) \geq w(b)$, and if $w(a) > w(b)$, then $v(a) > v(b)$, so we may assume $w(a) = w(b)$. By Lemma 5.4, for a fixed large n , there exist in R_n factorizations $a = \alpha \beta \tilde{a}$ and $b = \alpha \tilde{b}$, where $\text{ord}_n(\tilde{a}) = e(a)$ and $\text{ord}_n(\tilde{b}) = e(b)$. Let $\tilde{\tau} = \sum_{i=n}^\infty w(x_i)$. Theorem 5.10 implies that $w(\tilde{a}) = e(a)\tilde{\tau}$ and $w(\tilde{b}) = e(b)\tilde{\tau}$. Since $w(a) = w(b)$, $w(\tilde{a}) + w(\beta) = w(\tilde{b})$. Thus $e(a)\tilde{\tau} + w(\beta) = e(b)\tilde{\tau}$. Since $w(\beta) \geq 0$, it follows that $e(a) \leq e(b)$, so we conclude that $v(a) \geq v(b)$.

If in addition $a/b \in \mathfrak{M}_V$, we prove that $v(a) > v(b)$. Since $\text{ord}_n(a) > \text{ord}_n(b)$, $\text{ord}_n(\beta) > 0$. Thus $w(\beta) > 0$, so $e(a) < e(b)$, so $v(a) > v(b)$. \square

The following is immediate.

Corollary 6.5. *Assume Setting 3.3, and assume S is archimedean with $\dim S \geq 2$. Then the valuation domain W of Definition 6.2 is the rank 1 valuation overring of the boundary valuation ring V , and one of the following two statements holds.*

- (1) *There exist nonzero $a, b \in S$ such that $w(a) = w(b)$ and $e(a) \neq e(b)$. In this case V has rank 2.*
- (2) *For nonzero $a, b \in S$ with $w(a) = w(b)$, we have $e(a) = e(b)$. In this case $V = W$.*

7. The complete integral closure of an archimedean Shannon extension

Let S be a Shannon extension of R as in Setting 3.3. If S is non-archimedean, then the complete integral closure of S is $S^* = T$; see Theorem 8.3. The archimedean case is more subtle, and S may or may not be completely integrally closed. Theorem 7.1 describes the complete integral closure of an archimedean Shannon extension.

Theorem 7.1. *Assume Setting 3.3 and let w be as in Definition 6.2. Assume that S is archimedean with $\dim S \geq 2$. Let $y, a \in S$ be such that a is nonzero and yS is an N -primary ideal. Then the following are equivalent.*

- (1) $\frac{a}{y} \notin S$ and $\frac{a}{y}$ is almost integral over S .
- (2) $w(y) = w(a)$ and aS is not N -primary.
- (3) $N = (yS :_S a)$.

Moreover, every element of $S^* \setminus S$ has the form $\frac{a}{y}$ for some a, y with the stated properties.

Proof. Let V denote the boundary valuation ring of S .

(1) \Rightarrow (2) By Theorem 3.2(1), $S = V \cap T$, and by Theorem 3.7, $S^* = W \cap T$. Since $\frac{a}{y} \in T$ and $\frac{a}{y} \in S^* \setminus S$, it follows that $\frac{a}{y} \in W \setminus V$. Since $\frac{a}{y} \in W$, $w(a) \geq w(y)$. By Theorem 6.4, it follows that $w(a) = w(y)$ and $e(\frac{a}{y}) > 0$. Thus Lemma 5.2 implies aS is not N -primary.

(2) \Rightarrow (1) Since aS is not N -primary and yS is N -primary, by Lemma 5.2, $e(a) > 0$ and $e(y) = 0$. It follows from Theorem 6.4 that $\frac{a}{y} \notin V$, so $\frac{a}{y} \notin S$. Since $a \in S$ and $\frac{1}{y} \in T$, it follows that $\frac{a}{y} \in T$. Since $w(\frac{a}{y}) = 0$, $\frac{a}{y} \in W$. Thus $\frac{a}{y} \in W \cap T = S^*$.

(1) \Rightarrow (3) By Theorem 3.7, $(N :_F N)$ is the complete integral closure of S , and by assumption $\frac{a}{y}$ is almost integral over S . Hence $\frac{a}{y} \in N^{-1}$. Since N is a maximal ideal of S and $\frac{a}{y} \notin S$, it follows that $N = (yS :_S a)$.

(3) \Rightarrow (1) By assumption, $N = (yS :_S a)$. Hence $\frac{a}{y} \in (N :_F N) \setminus S$. Since $(N :_F N)$ is the complete integral closure of S , (1) follows.

Finally, we show every almost integral element $\frac{a}{y} \in S^* \setminus S$ with $a, y \in S$ has the property that yS is N -primary. Since $\frac{a}{y} \notin S$, we have $y \in N$. Since $S^* = W \cap T$, it follows that $\frac{a}{y} \in T$. By factoring out the common essential prime factors of a, y in R_n for $n \gg 0$ as in Proposition 4.4(3), we may assume that a, y have no common factors in the UFD T . Therefore $y \in T^\times$, so yS is N -primary. \square

Theorem 7.1 shows that the existence of almost integral elements depends on the existence of a pair of elements $a, y \in S$ with $w(y) = w(a)$ and $0 = e(y) < e(a)$. Using Theorem 5.10, we prove in Theorem 7.2 that S is completely integrally closed if and only if τ is rationally independent over the subgroup $w(T^\times)$ of \mathbb{R} .

Theorem 7.2. *Assume Setting 3.3 and let w and τ be as in Definition 6.2 and Notation 5.11. Assume that S is archimedean with $\dim S \geq 2$. Then the following are equivalent.*

- (1) S is not completely integrally closed.
- (2) $V \subsetneq W$.

(3) τ is rationally dependent over the subgroup $w(T^\times)$ of \mathbb{R} .⁷

Proof. (1) \Rightarrow (2) If $V = W$, then Theorem 3.7 implies S is completely integrally closed.

(2) \Rightarrow (3) Theorem 6.4 implies there exists an element $x \in F$ such that $w(x) = 0$ and $e(x) \neq 0$. Then Remark 6.3 implies τ is rationally dependent over $w(T^\times)$.

(3) \Rightarrow (1) By rational dependence, for some positive integer d there exists $c \in T^\times$ such that $w(c) = d\tau$. Since $\dim S \geq 2$, there exists $a \in S$ such that $e(a) > 0$ by Remark 5.3. By Remark 6.3, there is some $y \in T^\times$ such that $w(a) = w(y) + e(a)\tau$. Then

$$w(a^d) = dw(y) + de(a)\tau = w(y^d c^{e(a)}).$$

By Lemma 5.2(4), $a \notin T^\times$, thus a is not N -primary. By Theorem 7.1, $\frac{a^d}{y^d c^{e(a)}}$ is almost integral over S and not in S . \square

Remark 7.3. With notation as in Theorem 7.2, Theorem 6.4 implies that the condition $V \subsetneq W$ is equivalent to the condition that $\dim V = 2$.

Gilmer in [3, page 524] defines an integral domain A with quotient field K to be a *generalized Krull domain* if there is a set \mathcal{F} of rank 1 valuation overrings of A such that: (i) $A = \bigcap_{\mathcal{V} \in \mathcal{F}} \mathcal{V}$; (ii) for each $(\mathcal{V}, \mathfrak{M}_{\mathcal{V}}) \in \mathcal{F}$, we have $\mathcal{V} = A_{\mathfrak{M}_{\mathcal{V}} \cap A}$; and (iii) \mathcal{F} has finite character; that is, if $x \in K$ is nonzero, then x is a nonunit in only finitely many valuation rings of \mathcal{F} . The class of generalized Krull domains has been studied by a number of authors; see for example [7, 8, 12, 18, 19, 20].

Theorem 7.4. Assume Setting 3.3 and let w , W and τ be as in Notation 5.7, Definition 6.2 and Notation 5.11. If S is archimedean and not completely integrally closed, then

- (1) N is a height 1 prime of S^* , and $S_N^* = W$.
- (2) Every height 1 prime ideal of S^* is the radical of a principal ideal.
- (3) If $\tau \in w(T^\times)$, then every height 1 prime of S^* other than N is principal.
- (4) S^* is a generalized Krull domain.

Proof. By Theorem 3.7, N is the center of W on S^* , and S^* has the representation

$$S^* = W \cap T = W \cap \bigcap_{\substack{\mathfrak{p} \in \text{Spec } T \\ \text{ht } \mathfrak{p} = 1}} T_{\mathfrak{p}}.$$

⁷A real number $\lambda \in \mathbb{R}$ is rationally dependent over a subgroup $G \subseteq \mathbb{R}$ if and only if $d\lambda \in G$ for some positive integer d .

Since the representation of T as the intersection of its localizations at height 1 primes has finite character, so does this representation of S^* as the intersection of valuation domains. By [13, Lemma 2.3], the height 1 prime ideals of S^* are a subset of the contractions of the height 1 primes of T , along with possibly N . By [13, Lemma 1.1], to show $S_N^* = W$ and that N is a height 1 prime of S^* , it suffices to show that $\mathfrak{p} \cap S^* \not\subseteq N$ for each height 1 prime \mathfrak{p} of T .

Let $\mathfrak{p} = pT$ be a height 1 prime ideal of T . Then by Lemma 5.2(4), $e(p) > 0$. Theorem 7.2 implies that $d\tau = w(y)$ for some integer $d > 0$ and some N -primary element $y \in N$. By Remark 6.3, $w(p) = e(p)\tau + w(u)$ for some $u \in T^\times$. Denote $q = \frac{p^d}{y^{e(p)}u^d}$. Thus

$$\begin{aligned} w(q) &= w\left(\frac{p^d}{y^{e(p)}u^d}\right) \\ &= dw(p) - (e(p)w(y) + dw(u)) \\ &= d(\tau e(p) + w(u)) - (d\tau e(p) + dw(u)) \\ &= 0. \end{aligned}$$

Since $q \in \mathfrak{p}$ and q is a unit of W , $\mathfrak{p} \cap S^* \not\subseteq N$. This completes the proof of (1).

Furthermore, $\mathfrak{p} \cap S^* = \sqrt{qS^*}$. To see this, let $a \in \mathfrak{p} \cap S^*$. Since $a^d \in \mathfrak{p}^d = qT$, it follows that $\frac{a^d}{q} \in T$, and since $w(a) \geq 0$ and $w(q) = 0$, it follows that $\frac{a^d}{q} \in W$. Thus $\frac{a^d}{q} \in S^*$, so $a^d \in qS^*$. Since the only remaining height 1 prime ideal N of S^* is also the radical of a principal ideal, this completes the proof of (2).

If $\tau \in w(T^\times)$, we may take $q = \frac{p}{y^{e(p)}u}$. It follows that $\mathfrak{p} \cap S^* = qS^*$. This proves (3). Since S^* satisfies the conditions of a generalized Krull domain, (4) follows. \square

We indicate how to obtain completely integrally closed Shannon extensions that are not valuation domains. We use the following observations about rank 1 valuations that birationally dominate a sequence of local quadratic transforms.

Example 7.5. Let $\sigma > 2$ be an irrational real number. Starting from $R = R_0$, we inductively define a sequence of local quadratic transforms of 2-dimensional regular local rings R_i with regular system of parameters $\mathfrak{m}_i = (x_i, y_i)$,

$$(R_0, \mathfrak{m}_0) \subseteq \dots \subseteq (R_n, \mathfrak{m}_n).$$

We show that it is possible to choose the sequence R_i so that every rank 1 valuation ring \mathcal{V} that birationally dominates R_n has the following property: if we choose a valuation \mathfrak{v} for \mathcal{V} such that $\mathfrak{v}(x_0) = 1$, then it follows that $\sum_{i=0}^{n-1} \mathfrak{v}(\mathfrak{m}_i) = \lfloor \sigma \rfloor$ and $\frac{\sigma - \lfloor \sigma \rfloor}{\mathfrak{v}(x_n)} > 2$, where $\mathfrak{v}(x_n) = \frac{1}{2^e}$ for some integer $e \geq 2$.

Set $d = \lfloor \sigma \rfloor - 2$. Then for $0 \leq i < d$, set

$$R_{i+1} = R_i \left[\frac{y_i}{x_i} \right]_{(x_i, \frac{y_i}{x_i})}, \quad x_{i+1} = x_i, \quad y_{i+1} = \frac{y_i}{x_i}.$$

Then, set

$$R_{d+1} = R_d \left[\frac{y_d}{x_d} \right]_{(x_d, \frac{y_d}{x_d} - 1)}, \quad x_{d+1} = x_d, \quad y_{d+1} = \frac{y_d}{x_d} - 1.$$

In this construction, $x_{d+1} = x_0$ and $y_d = \frac{y_0}{x_0^d}$. Let \mathcal{V} be a rank 1 valuation birationally dominating R_{d+1} with $\mathfrak{v}(x_0) = 1$. By construction of R_{d+1} , $\mathfrak{v}(\frac{y_d}{x_d}) = 0$, so it follows that $\mathfrak{v}(y_d) = \mathfrak{v}(x_0) = 1$ and $\mathfrak{v}(y_0) = d + 1$. In particular,

$$\sum_{i=0}^d \mathfrak{v}(\mathfrak{m}_i) = \sum_{i=0}^d 1 = d + 1 = \lfloor \sigma \rfloor - 1.$$

Next, let e be an integer such that $2^e(\sigma - \lfloor \sigma \rfloor) > 2$. Then for $d + 1 \leq i < 2^e + d$, set

$$R_{i+1} = R_i \left[\frac{x_i}{y_i} \right]_{(y_i, \frac{x_i}{y_i})}, \quad x_{i+1} = \frac{x_i}{y_i}, \quad y_{i+1} = y_i.$$

Set $f = 2^e + d$ for convenience of notation. Then set

$$R_{f+1} = R_f \left[\frac{y_f}{x_f} \right]_{(x_f, \frac{y_f}{x_f} - 1)}, \quad x_{f+1} = x_f, \quad y_{f+1} = \frac{y_f}{x_f} - 1.$$

In this construction, $y_f = y_{d+1}$ and $x_f = \frac{x_{d+1}}{y_{d+1}^{2^e-1}}$. Let \mathcal{V} be a rank 1 valuation domain birationally dominating R_{f+1} with $\mathfrak{v}(x_0) = 1$. By construction of R_{f+1} , it follows that $\mathfrak{v}(\frac{x_f}{y_f}) = 0$, so $\mathfrak{v}(x_{d+1}) = 2^e \mathfrak{v}(y_{d+1})$. Since $\mathfrak{v}(x_{d+1}) = \mathfrak{v}(x_0) = 1$, it follows that $\mathfrak{v}(y_{d+1}) = \frac{1}{2^e} = \mathfrak{v}(x_{f+1})$. Therefore,

$$\sum_{i=d+1}^f \mathfrak{v}(\mathfrak{m}_i) = \sum_{i=d+1}^f \frac{1}{2^e} = (2^e) \frac{1}{2^e} = 1.$$

Set $n = f + 1$. We then have

$$\sum_{i=0}^{n-1} \mathfrak{v}(\mathfrak{m}_i) = \lfloor \sigma \rfloor.$$

and $\mathfrak{v}(x_n) = \frac{1}{2^e}$.

Example 7.6. Let $\sigma > 2$ be an irrational real number. We construct an example of an infinite sequence of 2-dimensional regular local rings (R_i, \mathfrak{m}_i) such that R_{i+1} is a local quadratic transform of R_i for each i , and $\sum_{i=0}^{\infty} \mathfrak{v}(\mathfrak{m}_i) = \sigma$ for the unique rank 1 valuation ring \mathcal{V} birationally dominating the union with $\mathfrak{v}(\mathfrak{m}_0) = 1$. Since $\dim R_i = 2$, we have $\mathcal{V} = \bigcup_{n=0}^{\infty} R_n$ by [1, Lemma 12, p. 337]. In addition, we show that the value group of \mathcal{V} is the additive group of $\mathbb{Z}[\frac{1}{2}]$. To construct this sequence, we repeat the construction in Example 7.5 an infinite number of times.

We start with $\sigma_0 = \sigma$ and $n_0 = 0$. Given σ_j and R_{n_j} , we construct the sequence of 2-dimensional regular local rings from R_{n_j} to $R_{n_{j+1}}$ for some $n_{j+1} > n_j$ as in Example 7.5. With this construction, for any valuation ring \mathcal{V} birationally dominating R_{n_j} with $\mathfrak{v}(\mathfrak{m}_{n_j}) = 1$, it follows that

$$\sum_{i=n_j}^{n_{j+1}-1} \mathfrak{v}(\mathfrak{m}_i) = \lfloor \sigma_j \rfloor.$$

Set $\sigma_{j+1} = 2^{e_j}(\sigma_j - \lfloor \sigma_j \rfloor)$, where e_j is as in Example 7.5.

We repeat this construction to obtain an infinite sequence of local quadratic transforms,

$$R_0 \subseteq \dots \subset R_{n_1} \subseteq \dots \subseteq R_{n_j} \subseteq \dots$$

Let $S = \bigcup_{n=0}^{\infty} R_n$. Then $S = \mathcal{V}$ is a valuation ring. Fix $\mathfrak{v}(\mathfrak{m}_0) = 1$. By Example 7.5 and by induction $\mathfrak{v}(\mathfrak{m}_{n_j}) = \prod_{i=0}^{j-1} \frac{1}{2^{e_i}}$ for every j . Therefore, for any j ,

$$\sum_{i=n_j}^{n_{j+1}-1} \mathfrak{v}(\mathfrak{m}_i) = \sigma_j \prod_{i=0}^{j-1} \frac{1}{2^{e_i}}.$$

A basic inductive argument yields that

$$\sigma - \sum_{i=0}^{n_j-1} \mathfrak{v}(\mathfrak{m}_i) < \prod_{i=0}^{j-1} \frac{1}{2^{e_i}}.$$

We conclude that $\sigma = \sum_{i=0}^{\infty} \mathfrak{v}(\mathfrak{m}_i)$. Furthermore, it follows that for every $i \geq 0$, $\mathfrak{v}(x_i), \mathfrak{v}(y_i) \in \frac{1}{2^k}\mathbb{Z}$ for some $k \geq 0$ (where k depends on i and increases as i increases).

Corollary 7.7. *There exists an archimedean Shannon extension S that is completely integrally closed but not a valuation domain.*

Proof. Let $\sigma > 2$ be an irrational real number. From the construction of Example 7.6, consider the sequence (R_i, \mathfrak{m}_i) of local quadratic transforms of 2-dimensional regular local rings. Let \mathfrak{v} be the rank 1 valuation for $\mathcal{V} = \bigcup_{i=0}^{\infty} R_i$ such that

$\mathfrak{v}(\mathfrak{m}_0) = 1$, so that $\sum_{i=0}^{\infty} \mathfrak{v}(\mathfrak{m}_i) = \sigma$ and the value group of \mathfrak{v} is the additive group of $\mathbb{Z}[\frac{1}{2}]$.

Let z be an indeterminate, let $x_n R_{n+1} = \mathfrak{m}_n R_{n+1}$ for $n \geq 0$, and denote $z_n = \frac{z}{\prod_{i=0}^{n-1} x_i}$. Then consider the sequence of local quadratic transforms of 3-dimensional regular local rings, $R'_i = R_i[z_i]_{(\mathfrak{m}_i, z_i)}$. Since $e(z) = 1$, Remark 5.3 implies the Shannon extension $S = \bigcup_{i=0}^{\infty} R'_i$ is not a rank 1 valuation ring. Let w be as in Notation 5.7 with $w(\mathfrak{m}_0) = 1$. Notice that for $a \in R_i$, we have $\text{ord}_{R_i}(a) = \text{ord}_{R'_i}(a)$, so the restriction of w to the quotient field of \mathcal{V} is equal to \mathfrak{v} . Thus

$$\sum_{n=0}^{\infty} w(\mathfrak{m}_n) = \sigma, \quad w(T^\times) = \bigcup_{n=0}^{\infty} \frac{1}{2^n} \mathbb{Z}.$$

Since $\sigma = \tau < \infty$, Theorem 6.1 implies that S is archimedean. Since σ is not rationally dependent over $w(T^\times)$, Theorem 7.2 implies that S is completely integrally closed. \square

8. Non-archimedean Shannon extensions

In this section we describe the Shannon extensions that are not archimedean. Like the archimedean case, the functions $e : F^\times \rightarrow \mathbb{Z}$ and $w : F \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ from Definition 5.1 and Notation 5.7 describe the boundary valuation v in terms of a composite. In the archimedean case, w defines a valuation ring W on F , and if $S \neq S^*$, then e defines a valuation on the residue field of W . In the non-archimedean case, the situation reverses: e defines a valuation ring E on F and w defines a valuation on the residue field of E .

Theorem 8.1. *Assume Setting 3.3 and assume that S is non-archimedean. For $a \in F^\times$, we have:*

- (1) *If $e(a) > 0$, then $w(a) = +\infty$.*
- (2) *The set $P_\infty = \{a \in S \mid w(a) = +\infty\}$ is a prime ideal of both S and T .*

Proof. To prove item 1, we proceed as in the proof of Theorem 5.10. Let $a \in F^\times$ and assume that $e(a) > 0$. Write $a = \frac{b}{c}$ for some $b, c \in N$. There exist for a fixed large integer m factorizations $b = u\tilde{b}$ and $c = v\tilde{c}$ in R_m as in Lemma 5.4, where $u, v \in T^\times$, $\text{ord}_m(\tilde{b}) = e(\tilde{b}) = e(b)$, and $\text{ord}_m(\tilde{c}) = e(\tilde{c}) = e(c)$. That is, fix m sufficiently large so that the orders of the transforms of the principal ideals $\tilde{b}R_m$ and $\tilde{c}R_m$ are constant, namely $e(b)$ and $e(c)$, respectively. By Lemma 5.2(4), $e(u) = e(v) = 0$, so by Theorem 5.6, $w(u)$ and $w(v)$ exist and are finite. Thus to prove that $w(a) = +\infty$ it suffices to show that $w(\frac{\tilde{b}}{\tilde{c}}) = +\infty$.

For $n \geq 0$, we have $\mathfrak{m}_n R_{n+1} = x_n R_{n+1}$, and by the assumptions of Setting 3.3, $x_n \in T^\times$. For $n \geq m$, Remark 2.3 implies that

$$\tilde{b} = \left(\prod_{i=m}^{n-1} x_i^{e(b)} \right) b_n, \quad \tilde{c} = \left(\prod_{i=m}^{n-1} x_i^{e(c)} \right) c_n, \quad \frac{\tilde{b}}{\tilde{c}} = \left(\prod_{i=m}^{n-1} x_i^{e(a)} \right) \frac{b_n}{c_n}$$

for some $b_n, c_n \in R_n$ with $\text{ord}_n(b_n) = e(b)$ and $\text{ord}_n(c_n) = e(c)$. Thus for all $n \geq m$ and for all $j \geq 0$,

$$\text{ord}_j \left(\frac{\tilde{b}}{\tilde{c}} \right) = \text{ord}_j \left(\frac{b_n}{c_n} \right) + e(a) \sum_{i=m}^{n-1} \text{ord}_j(x_i).$$

Dividing both sides by $\text{ord}_j(x)$,

$$\frac{\text{ord}_j \left(\frac{\tilde{b}}{\tilde{c}} \right)}{\text{ord}_j(x)} = \frac{\text{ord}_j \left(\frac{b_n}{c_n} \right)}{\text{ord}_j(x)} + e(a) \sum_{i=m}^{n-1} \frac{\text{ord}_j(x_i)}{\text{ord}_j(x)}.$$

Taking the limit as $j \rightarrow \infty$ and applying Theorem 5.6,

$$w \left(\frac{\tilde{b}}{\tilde{c}} \right) = w \left(\frac{b_n}{c_n} \right) + e(a) \sum_{i=m}^{n-1} w(x_i).$$

Furthermore, for $j \geq n \geq m$,

$$\frac{b_n}{c_n} = \left(\prod_{i=n}^{j-1} x_i^{e(a)} \right) \frac{b_j}{c_j}, \quad \text{ord}_j \left(\frac{b_n}{c_n} \right) = e(a) + e(a) \sum_{i=n}^{j-1} \text{ord}_j(x_i) > 0,$$

so $\text{ord}_j \left(\frac{b_n}{c_n} \right) > 0$. Therefore $w \left(\frac{b_n}{c_n} \right) \geq 0$. It follows that $w \left(\frac{\tilde{b}}{\tilde{c}} \right) \geq e(a) \sum_{i=m}^{n-1} w(x_i)$ for all $n \geq m$. Theorem 6.1 implies that $\sum_{i=m}^{\infty} w(x_i) = +\infty$, so it follows that $w \left(\frac{\tilde{b}}{\tilde{c}} \right) = +\infty$ and thus $w(a) = +\infty$. This establishes item 1.

That P_∞ is a prime ideal of S follows from Remark 5.8. Since $T = S[1/x]$ and $w(x) = 1$, it follows from Remark 5.8 that T has no elements of w -value $-\infty$ and that $P'_\infty = \{a \in T \mid w(a) = +\infty\}$ is a prime ideal of T . By Theorem 5.9(2), we have $P'_\infty \subseteq V$. Since $S = T \cap V$ by Theorem 3.2, we conclude that $P_\infty = P'_\infty$. This establishes item 2. \square

Remark 8.2. With notation as in Theorem 8.1, the prime ideal

$$P_\infty = \{a \in N \mid aS \text{ is not } N\text{-primary}\}.$$

It follows that P_∞ is the unique prime ideal of S of dimension 1, and we have $T = S_{P_\infty} = S[1/x]$, where T is the Noetherian hull of S . If $a \in N \setminus P_\infty$ and $b \in P_\infty$, then $\frac{b}{a} \in P_\infty \subseteq S$. Hence $b \in aS$.

In Theorem 8.3 we characterize among Shannon extensions of dimension at least 2 those that are non-archimedean.⁸

Theorem 8.3. *Assume Setting 3.3 and that $\dim S \geq 2$. Let $P = \bigcap_{n \geq 1} x^n S$. Then the following are equivalent.*

- (1) S is non-archimedean.
- (2) $S^* = T = (P : P)$.
- (3) P is a nonzero prime ideal of S .
- (4) Every nonmaximal prime ideal of S is contained in P .
- (5) T is a regular local ring.

Proof. The equivalence of the first three items is established in [11, Theorem 6.9 and Corollary 6.10].

(1) \Rightarrow (4) Since $T = S[1/x]$, we have $P = (S : T)$. Since S is not archimedean, Theorem 8.1(2) gives that P_∞ is an ideal of T . Thus $P_\infty \subseteq (S : T) = P$. By Remark 8.2, every nonmaximal prime ideal of R is contained in P_∞ , so this forces $P_\infty = P$. Hence (4) holds.

(4) \Rightarrow (5) Since $T = S[1/x]$ and x is an N -primary element of S , (4) implies that PT is the unique maximal ideal of T . Thus by Theorem 3.2, T is a regular local ring.

(5) \Rightarrow (1) Suppose S is archimedean. By assumption $\dim S > 1$. Thus there exists $f \in S$ such that f is a non-unit of T , so by Lemma 5.2(4), $e(f) > 0$. Since S is archimedean, there exists an N -primary element $y \in S$ such that $w(y) > w(f)$. With notation as in Theorem 6.4, since $v(y) > v(f)$, it follows that $v(f) = v(f + y)$, so $e(f + y) = e(f)$. Thus by Lemma 5.2(4), $f + y$ is a non-unit of T . But $(f + y) - f = y$ is a unit of T . Therefore T is not local. \square

Corollary 8.4. *Assume Setting 3.3. Assume that S is non-archimedean with $\dim S \geq 2$ and denote $P = \bigcap_{n \geq 1} x^n S$. Then S/P is a rank 1 valuation domain on the residue field T/P of T . Every valuation domain \mathcal{V} that dominates S has a prime ideal lying over P .*

Proof. The principal N -primary ideals of S are linearly ordered with respect to inclusion [11, Corollary 5.5], each principal N -primary ideal contains P , and S/P is a 1-dimensional local domain by Theorem 8.3. Thus S/P is a rank 1 valuation domain.

Let \mathcal{V} be a valuation domain dominating S and let $Q = \bigcap_{n \geq 1} x^n \mathcal{V}$. Then Q is a prime ideal of \mathcal{V} and $x \notin Q$. Since $x^n S \subseteq x^n \mathcal{V}$, we have $P \subseteq Q$. Also, since $x \notin Q$,

⁸A non-archimedean integral domain necessarily has dimension at least 2 by Remark 3.6.

$Q \cap S$ is a nonmaximal prime ideal of S . By Theorem 8.3, every nonmaximal prime ideal of S is contained in P , so we conclude that $P = Q \cap S$. \square

In Theorem 8.5 and Corollary 8.6, we give a complete description of the boundary valuation in the non-archimedean case.

Theorem 8.5. *Assume Setting 3.3 and let e and w be as in Definition 5.1 and Notation 5.7. Assume that S is non-archimedean with $\dim S \geq 2$ and denote $P = \bigcap_{n \geq 1} x^n S$. Then:*

- (1) *e is a rank 1 valuation on F whose valuation ring E contains V . If in addition $R/(P \cap R)$ is a regular local ring, then E is the order valuation ring of T .*
- (2) *w induces a rational rank 1 valuation w' on the residue field E/\mathfrak{m}_E of E . The valuation ring W' defined by w' extends the valuation ring S/P , and the value group of W' is the same as the value group of S/P .*
- (3) *V is the valuation ring defined by the composite valuation of e and w' .*

The following pullback diagram illustrates V :

$$\begin{array}{ccc} V & \longrightarrow & V/\mathfrak{m}_E = W' \\ \downarrow & & \downarrow \\ E & \longrightarrow & E/\mathfrak{m}_E \end{array}$$

Proof. We prove that the multiplicative function e defines the rank 1 valuation overring of V . By Theorem 8.1, if $a \in F^\times$ and $e(a) > 0$, then $w(a) = +\infty$, so $a \in \mathfrak{m}_V$ by Theorem 5.9(2). On the other hand, if $a \in F^\times$ and $e(a) < 0$, then $w(a) = -\infty$, so $a \notin V$. Therefore if $a \in V$ is nonzero, then $e(a) \geq 0$.

To see that e defines a valuation, it suffices to show that nonzero elements of F with positive e -value are closed under addition. Let $a, b \in F$ be two such elements, and assume without loss of generality that $(a, b)V = aV$. Thus $\frac{b}{a} \in V$, so $e(b) \geq e(a)$, and $1 + \frac{b}{a} \in V$, so $e(1 + \frac{b}{a}) \geq 0$. Therefore $e(a + b) = e(a(1 + \frac{b}{a})) = e(a) + e(1 + \frac{b}{a}) \geq e(a)$. It follows that e defines a rank 1 discrete valuation ring E , and the maximal ideal \mathfrak{m}_E of E is a prime ideal of V . Thus $V_{\mathfrak{m}_E} = E$; cf. [17, (11.3)].

Assume in addition that $R/(P \cap R)$ is a regular local ring. Let $\mathfrak{p}_i = P \cap R_i$. Since S is non-archimedean, Theorem 6.1 implies that $\sum_{i=0}^{\infty} w'(\mathfrak{m}_i/\mathfrak{p}_i) = \infty$. Thus [6, Theorem 10] implies that for every element $f \in R_i$ such that $\text{ord}_{R_i}(f) = e(f)$, we have that $\text{ord}_{(R_i)_{\mathfrak{p}_i}}(f) = \text{ord}_{R_i}(f)$. Since $(R_i)_{\mathfrak{p}_i} = T$, we conclude E is the order valuation ring of T . This completes the proof of item 1.

For items 2 and 3, we first observe that by Remark 5.8, w induces a rank 1 valuation w' on E/\mathfrak{m}_E . We prove that $V = \{a \in F \mid w(a) \geq 0\}$. For $a \in F$ such that $w(a) \neq 0$, it follows from the definitions of V and w that $a \in V$ if and only if

$w(a) > 0$. For $a \in F$ such that $w(a) = 0$, Theorem 8.1 implies that $e(a) = 0$, so Theorem 5.6(5) implies that $a \in V$. This proves item 3.

From Proposition 5.12, it follows that w' has rational rank 1. Since $P = \mathfrak{m}_E \cap S$ from Theorem 8.1, the valuation ring W' extends the valuation ring S/P . Corollary 5.5 implies that the range of w' is the same as the range of its restriction to the field T/P . This completes the proof of item 2. \square

Corollary 8.6 describes a valuation v on the field F that defines the boundary valuation V of the Shannon extension in Theorem 8.5.

Corollary 8.6. *Assume notation as in Theorem 8.5. The boundary valuation V of S has rank 2 and rational rank 2. Fix an element $z \in E$ such that $\mathfrak{m}_E = zE$. For an element $a \in F^\times$, define*

$$v : F^\times \rightarrow \mathbb{Z} \oplus \mathbb{Q}$$

$$a \mapsto \left(\frac{e(a)}{e(z)}, w \left(\frac{a}{z^{\frac{e(a)}{e(z)}}} \right) \right),$$

where $\mathbb{Z} \oplus \mathbb{Q}$ is ordered lexicographically. The function v is a valuation on F that defines the boundary valuation ring V .

Proof. Theorem 8.5(4) states that $V/\mathfrak{m}_E = W'$ and $V_{\mathfrak{m}_E} = E$. Since $\mathfrak{m}_E = zE$ and e is a valuation defining E , the fraction $\frac{e(a)}{e(z)}$ is an integer and $aE = z^{\frac{e(a)}{e(z)}}E$. Therefore a and $z^{\frac{e(a)}{e(z)}}$ have the same e -value and their ratio has e -value zero, and hence has finite w -value. Theorem 8.5(3) says w' has rational rank 1. Thus $v(F^\times) \subseteq \mathbb{Z} \oplus \mathbb{Q}$. The function v is a homomorphism of the multiplicative group F^\times into the additive group $\mathbb{Z} \oplus \mathbb{Q}$, and for $a \in F^\times$, we have $v(a) \geq 0 \iff a \in V$. It follows that v is a valuation on F that defines V . \square

Unlike the archimedean case considered in Theorem 6.4, the boundary valuation in the non-archimedean case defined in Corollary 8.6 depends on assigning a value in the second component to an element $z \in E$ of minimal positive e -value.

Theorem 8.5 implies that a non-archimedean Shannon extension may be described as a pullback. In the paper [14] in preparation, we are interested in characterizing non-archimedean Shannon extensions as pullbacks.

References

- [1] S. Abhyankar, On the valuations centered in a local domain. Amer. J. Math. 78 (1956), 321–348.

- [2] N. Bourbaki, *Commutative Algebra* Chapters 1-7, Springer-Verlag, Berlin, 1989.
- [3] R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker, New York, 1972.
- [4] A. Granja, Valuations determined by quadratic transforms of a regular ring. *J. Algebra* 280 (2004), no. 2, 699–718.
- [5] A. Granja, M. C. Martinez and C. Rodriguez, Valuations with preassigned proximity relations, *J. Pure Appl. Algebra* 212 (2008), 1347–1366.
- [6] A. Granja and T. Sánchez-Giralda. Valuations, equimultiplicity and normal flatness. *J. Pure Appl. Algebra* 213 (2009), no. 9, 1890–1900.
- [7] M. Griffin, Rings of Krull type, *J. reine angew. Math.* 229 (1968), 1–27.
- [8] M. Griffin, Families of finite character and essential valuations, *Trans. Amer. Math. Soc.* 130 (1968), 75–85.
- [9] W. Heinzer, M.-K. Kim and M. Toeniskoetter, Finitely supported $*$ -simple complete ideals in a regular local ring, *J. Algebra* 401 (2014), 76–106.
- [10] W. Heinzer, M.-K. Kim and M. Toeniskoetter, Directed unions of local quadratic transforms of a regular local ring, preprint.
- [11] W. Heinzer, K. A. Loper, B. Olberding, H. Schoutens and M. Toeniskoetter, Ideal Theory of Infinite Directed Unions of Local Quadratic Transforms, arXiv:1505.06445.
- [12] W. Heinzer and J. Ohm, Defining families for integral domains of real finite character, *Canad. J. Math.* **24** (1972), 1170–1177.
- [13] W. Heinzer and J. Ohm, Noetherian intersections of integral domains. *Trans. Amer. Math. Soc.* 167 (1972), pp. 291–308.
- [14] W. Heinzer, B. Olberding, and M. Toeniskoetter, Directed unions of regular local rings as pullbacks, in preparation.
- [15] J. Lipman, On complete ideals in regular local rings. *Algebraic geometry and commutative algebra*, Vol. I, 203–231, Kinokuniya, Tokyo, 1988.
- [16] H. Matsumura *Commutative Ring Theory* Cambridge Univ. Press, Cambridge, 1986.
- [17] M. Nagata, *Local Rings*, John Wiley, New York, 1962.

- [18] E. Paran and M. Temkin, Power series over generalized Krull domains. *J. Algebra* 323 (2010), no. 2, 546–550.
- [19] E. Pirtle, Families of valuations and semigroups of fractionary ideal classes, *Trans. Amer. Math. Soc.* 144 (1969), 427–439.
- [20] P. Ribenboim, Le théorème d’approximation pour les valuations de Krull, *Math. Zeit.* 68 (1957/58), 1–18.
- [21] D. Shannon, Monoidal transforms of regular local rings. *Amer. J. Math.* 95 (1973), 294–320.